

# THE BERNSTEIN INEQUALITY FOR SLICE REGULAR POLYNOMIALS

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**ABSTRACT.** In this paper, we give an alternative proof of the Bernstein inequality for slice regular polynomials. We also establish the Erdős-Lax inequality for a subclass of quaternion polynomials in terms of the convex combination identity for slice regular functions. Finally, we extend a result of Ankeny-Rivlin to the quaternionic setting via the Hopf lemma.

## 1. THE BERNSTEIN INEQUALITY

Recently, the classical Bernstein inequality has been extended to the quaternionic setting depending heavily on the Gauss-Lucas type theorem and the theory of zero sets of quaternionic slice regular polynomials in [20]. In this paper, we give an alternative proof of the Bernstein inequality in the setting of quaternions, octonions and Clifford algebra applying the Fejér kernel. For more extensions of the Bernstein inequality for the complex polynomials, we refer to the books [3, Chapter 1], [6, Chapter 4], [12, Chapter 2] and [13, Chapter 14].

Now we recall some preliminary definitions and results for quaternions.

Let  $\mathbb{H}$  denote the noncommutative, associative, real algebra of quaternions with standard basis  $\{1, i, j, k\}$ , subject to the multiplication rules

$$i^2 = j^2 = k^2 = ijk = -1.$$

Every element  $q = x_0 + x_1i + x_2j + x_3k = x_0 + \underline{q}$  in  $\mathbb{H}$  is composed by the *real* part  $\operatorname{Re}(q) = x_0$  and the *imaginary* part  $\operatorname{Im}(q) = x_1i + x_2j + x_3k$ . The *conjugate* of  $q \in \mathbb{H}$  is then  $\bar{q} = \operatorname{Re}(q) - \operatorname{Im}(q)$  and its *modulus* is defined by  $|q| = \sqrt{q\bar{q}} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$ . The inverse of each nonzero element  $q$  of  $\mathbb{H}$  is given by  $q^{-1} = |q|^{-2}\bar{q}$ . Every  $q \in \mathbb{H}$  can be expressed as  $q = x + yI$ , where  $x, y \in \mathbb{R}$  and

$$I = \frac{\operatorname{Im}(q)}{|\operatorname{Im}(q)|}$$

if  $\operatorname{Im} q \neq 0$ , otherwise we take  $I$  arbitrarily such that  $I^2 = -1$ . Here  $I$  is an element of the unit 2-sphere of purely imaginary quaternions,

$$\mathbb{S} = \{q \in \mathbb{H} : q^2 = -1\}.$$

For every  $I \in \mathbb{S}$ , we denote by  $\mathbb{C}_I$  the plane  $\mathbb{R} \oplus I\mathbb{R}$ , isomorphic to  $\mathbb{C}$ , and by  $\mathbb{B}_I$  the intersection  $\mathbb{B} \cap \mathbb{C}_I$ , where  $\mathbb{B} = \{q \in \mathbb{H} : |q| < 1\}$ .

For any slice regular polynomial  $P(q) = \sum_{j=0}^n q^j a_j$  with quaternionic coefficients  $a_j$ , its derivatives is defined as

$$P'(q) = \sum_{j=1}^n q^{j-1} j a_j.$$

Let  $P_I$  be the restriction of  $P$  to  $\mathbb{B}_I$  and denote  $\|P\| = \max_{|q| \leq 1} |P(q)|$ .

With above notations, we now can present the main result as follows.

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**Theorem 1.1.** *Let  $P(q) = \sum_{j=0}^n q^j a_j$  be a slice regular polynomial of degree  $n$  with quaternionic coefficients. Then*

$$(1.1) \quad \|P'\| \leq n\|P\|.$$

*Moreover, the equality holds if and only if  $P(q) = q^n a_n$  for some  $a_n \in \mathbb{H}$ .*

To prove this theorem, we resort to the Fejér kernel. The Fejér kernel is defined as

$$F_n(x) = \frac{1}{n+1} \sum_{k=0}^n D_k(x),$$

where

$$D_k(x) = \sum_{s=-k}^k e^{isx}$$

is the  $k$ -th order Dirichlet kernel.

It can also be written in a closed form as

$$F_n(x) = \frac{1}{n+1} \left( \frac{\sin \frac{(n+1)x}{2}}{\sin \frac{x}{2}} \right)^2 = \frac{1}{n+1} \frac{1 - \cos(n+1)x}{1 - \cos x}.$$

The important feature of the Fejér kernel is its non-negative. The Fejér kernel can also be expressed as

$$(1.2) \quad F_n(x) = \sum_{j=-n}^n \left( 1 - \frac{|j|}{n+1} \right) e^{ijx}.$$

Let  $g : \mathbb{R} \rightarrow \mathbb{H}$  be continuous and  $2\pi$ -periodic. Consider its Fourier series

$$g(\theta) = \sum_{j=-\infty}^{+\infty} e^{ij\theta} c_j, \quad c_j = \frac{1}{2\pi} \int_0^{2\pi} e^{-ij\theta} g(\theta) d\theta.$$

The  $n$ -th partial sum of the Fourier series is given by

$$s_n(\theta; g) = \sum_{j=-n}^n e^{ij\theta} c_j = \frac{1}{2\pi} \int_0^{2\pi} D_n(\theta - \varphi) g(\varphi) d\varphi$$

and the corresponding  $n$ -th Cesàro sum has the expression

$$(1.3) \quad \sigma_n(\theta; g) = \frac{1}{n+1} \sum_{j=0}^n s_j(\theta; g) = \frac{1}{2\pi} \int_0^{2\pi} F_n(\theta - \varphi) g(\varphi) d\varphi.$$

Form (1.2), we thus have

$$(1.4) \quad \sigma_n(\theta; g) = \sum_{j=-n}^n \left( 1 - \frac{|j|}{n+1} \right) e^{ij\theta} c_j.$$

for any  $g(\theta) = \sum_{j=-\infty}^{+\infty} e^{ij\theta} c_j$ .

With above preliminaries, we come to give the proof of Theorem 1.1.

*Proof of Theorem 1.1.* Starting from the slice regular polynomial  $P(q) = \sum_{j=0}^n q^j a_j$ , for any fixed  $I \in \mathbb{S}$ , we introduce a continuous and  $2\pi$ -periodic function  $g : \mathbb{R} \rightarrow \mathbb{H}$  via

$$g(\theta) = e^{In\theta} P(e^{-I\theta}).$$

It is evident that

$$(1.5) \quad \max_{\theta \in \mathbb{R}} |g(\theta)| = \max_{q \in \mathbb{B}_I} |P_I(q)|,$$

and

$$g(\theta) = \sum_{j=0}^n e^{Ij\theta} a_{n-j}.$$

Notice that

$$(1.6) \quad \left. \frac{1}{n} q^{-(n-1)} P'(q) \right|_{q=e^{I\theta}} = \sum_{j=0}^n \frac{j}{n} e^{(j-n)I\theta} a_j = \sum_{j=0}^{n-1} \left(1 - \frac{j}{n}\right) e^{-Ij\theta} a_{n-j}.$$

The right side is equal to  $\sigma_{n-1}(-\theta; g)$  due to (1.4).

From the inequality (1.3), it follows that

$$(1.7) \quad \sigma_{n-1}(-\theta; g) = \frac{1}{2\pi} \int_0^{2\pi} F_{n-1}(-\theta - \varphi) g(\varphi) d\varphi.$$

Combing this with (1.6), we have

$$(1.8) \quad \left| \frac{1}{n} q^{-(n-1)} P'(q) \right|_{q=e^{I\theta}} \leq \max_{|q|=1} |P_I(q)|, \quad \forall I \in \mathbb{S}, \theta \in [0, 2\pi),$$

which implies that

$$|P'(q)| \leq n \max_{|q|=1} |P(q)|, \quad \forall q \in \partial \mathbb{B}.$$

That is

$$\max_{|q|=1} |P'(q)| \leq n \max_{|q|=1} |P(q)|.$$

From the maximum modulus principle for slice regular functions [9, Theorem 7.1], we can get that

$$\max_{|q|=1} |P(q)| = \max_{|q| \leq 1} |P(q)| = \|P\|,$$

which leads to the inequality (1.1).

When equality occurs in (1.1), we have

$$|P'(q_0)| = n \max_{|q|=1} |P(q)|,$$

for some  $q_0 = e^{I_0\theta_0} \in \partial \mathbb{B}$  with  $I_0 \in \mathbb{S}$  and  $\theta_0 \in [0, 2\pi)$ . From (1.5), (1.7) and (1.8), we see that the function  $g$  is a constant saying  $a_n$ . By the identity principle for slice regular functions (see e.g., [9, Theorem 1.12]), it holds that  $P(q) = q^n a_n$ , as desired.  $\square$

Some useful remarks concerning Theorem 1.1 are in order.

**Remark 1.2.**

(I) In the proof of Theorem 1.1 above, as a byproduct, we get the following:

Let  $P(q) = \sum_{j=0}^n q^j a_j$  be a slice regular polynomial of degree  $n$  with quaternionic coefficients. Then

$$\min_{|q|=1} |P'(q)| \geq n \min_{|q|=1} |P(q)|.$$

Moreover, the equality holds if and only if  $P(q) = q^n a_n$  for some  $a_n \in \mathbb{H}$ .

(II) Taking the same process as in Theorem 1.1, we can get the following Bernstein inequality in the Clifford algebra setting by the maximum modulus principle for slice monogenic functions [15, Theorem 3.1] and Lemma 1.4 below. For precise definition of slice monogenic functions, we refer to [5].

**Theorem 1.3.** *Let  $P(x) = \sum_{j=0}^n x^j a_j$  be a (left) slice monogenic polynomial of degree  $n$  with Clifford algebra coefficients  $a_n \in \mathbb{R}_{0,m}$ . Then*

$$(1.9) \quad \|P'\| \leq n\|P\|,$$

where the norm of  $P$  is defined by  $\|P\| = \max_{|x| \leq 1} |P(x)|$ .

Moreover, the equality holds if and only if  $P(x) = x^n a_n$  for some  $a_n \in \mathbb{R}_{0,m}$ .

**Lemma 1.4.** [10, Theorem 3.14] *For any  $a, b \in \mathbb{R}_{0,m}$  with  $b\bar{b} = |b|^2$ , we have*

$$|ab| = |a||b|.$$

In particular,

$$|xb| = |x||b|$$

for any  $b \in \mathbb{R}_{0,m}$  and any paravector  $x \in \mathbb{R}^{m+1}$ .

(III) The nonassociative nature of octonions plays no role in the proof of Theorem 1.1 since Artin's theorem (cf. [19]) implies that the subalgebra generated by two elements in octonions is associative. Hence Theorem 1.1 still holds for octonionic slice regular polynomials in the sense of Gentili and Struppa in [8].

(IV) For the quaternionic Hardy space  $H^p(\mathbb{B})$  ( $1 \leq p < +\infty$ ) introduced in [18], that is, the Banach space of slice regular functions  $f$  on  $\mathbb{B}$  with the norm

$$\|f\|_p := \sup_{I \in \mathbb{S}} \lim_{r \rightarrow 1^-} \left( \int_0^{2\pi} |f_I(re^{I\theta})|^p d\theta \right)^{\frac{1}{p}} < +\infty.$$

In fact, Theorem 1.1 is a version of the limit case ( $p = +\infty$ ). And it also holds for the case of  $p = 2$  as follows.

**Theorem 1.5.** *Let  $P(q) = \sum_{j=0}^n q^j a_j$  be a slice regular polynomial of degree  $n$  with quaternionic coefficients. Then*

$$\|P'\|_2 \leq n\|P\|_2.$$

Moreover, the equality holds if and only if  $P(q) = q^n a_n$  for some  $a_n \in \mathbb{H}$ .

*Proof.* For any  $I \in \mathbb{S}$ , it holds that

$$\int_0^{2\pi} |P_I(e^{I\theta})|^2 d\theta = \int_0^{2\pi} \overline{P_I(z)} P_I(z) d\theta = \int_0^{2\pi} \sum_{m=0}^n \bar{a}_m e^{-mI\theta} \sum_{m=0}^n e^{mI\theta} a_m d\theta = \sum_{m=0}^n |a_m|^2.$$

Hence we have that

$$\int_0^{2\pi} |P'_I(e^{I\theta})|^2 d\theta = \sum_{m=0}^n m^2 |a_m|^2 \leq n^2 \sum_{m=0}^n |a_m|^2,$$

as desired.  $\square$

## 2. THE ERDÖS-LAX INEQUALITY

The refinement of the Bernstein inequality conjectured by Erdős and proved by Lax [11], stating that

$$(2.1) \quad \|p'\| \leq \frac{n}{2} \|p\|$$

for those complex polynomials  $p$  of degree  $n$  that has no zero in the open unit disk of  $\mathbb{C}$ . For other proofs of this inequality see [2] and references therein.

As pointed out in [20, Theorem 3.1], this result fails in the quaternionic setting in general. See the counterexample  $P(q) = (q - i) * (q - j)$ . However, the Erdős-Lax inequality holds true for a subclass of the quaternionic polynomials of degree  $n$  as follows.

**Proposition 2.1.** *Let  $P(q)$  be a slice regular polynomial of degree  $n$  with quaternionic coefficients that has no zero in the ball  $\mathbb{B}$ . Assume that the zeros of  $P(q)$  are either spheres and/or real points and that there exists at most one isolated zero  $\alpha \in \mathbb{H} \setminus \mathbb{R}$  that has multiplicity 1. Then*

$$\|P'\| \leq \frac{n}{2} \|P\|.$$

Now we extend this result to the following version using the convex combination identity for slice regular functions [14].

**Proposition 2.2.** *Let  $P(q) = \sum_{j=0}^n q^j a_j$  be a slice regular polynomial of degree  $n$  with all coefficients  $a_j \in \mathbb{H}$ . Assume that, for some  $I \in \mathbb{S}$ ,  $P_I$  has no zero in the open unit disk  $\mathbb{B}_I$  in  $\mathbb{C}_I$ ,  $P_I(\mathbb{C}_I) \subset \mathbb{C}_I$ . Then*

$$\|P'\| \leq \frac{n}{2} \|P\|.$$

*Proof.* Let  $P = \sum_{j=0}^n q^j a_j$  be the polynomial as described in the proposition. Notice that  $P_I(\mathbb{C}_I) \subset \mathbb{C}_I$  for some  $I \in \mathbb{S}$ , equivalently,  $a_j \in \mathbb{C}_I$  for  $I \in \mathbb{S}$ , and  $j = 0, 1, \dots, n$ , which implies that  $P'_I(\mathbb{C}_I) \subset \mathbb{C}_I$ . The classical Erdős-Lax inequality applying to  $P_I$  yields that

$$(2.2) \quad \|P'_I\| \leq \frac{n}{2} \|P_I\|,$$

where  $\|P_I\| = \max_{q \in \mathbb{B}_I} |P_I(q)|$ .

Using the convex combination identity for slice regular functions [14, Lemma 4.4], it holds that

$$(2.3) \quad |P'(\alpha + \beta J)|^2 = \frac{1 + \langle J, I \rangle}{2} |P'(\alpha + \beta I)|^2 + \frac{1 - \langle J, I \rangle}{2} |P'(\alpha - \beta I)|^2,$$

where  $\langle \cdot, \cdot \rangle$  is Euclidean inner product in  $\mathbb{R}^3$ .

From (2.2) and (2.3), we have

$$|P'(\alpha + \beta J)| \leq \frac{n}{2} \|P_I\| \leq \frac{n}{2} \|P\|, \quad \forall \alpha + \beta J \in \mathbb{B}_J, \quad \forall J \in \mathbb{S},$$

proving the claim.  $\square$

### 3. A RESULT OF ANKENY-RIVLIN

As an application of the Erdős-Lax inequality, Ankeny and Rivlin show the following in [1].

**Theorem 3.1.** *If  $p(z)$  is a complex polynomial of degree  $n$  such that  $\max_{|z| \leq 1} |p(z)| = 1$ , and  $p(z)$  has no zero within the unit circle, then*

$$\max_{|z|=R>1} |p(z)| \leq \frac{1 + R^n}{2},$$

with equality only for  $p(z) = (\lambda + \mu z^n)/2$ , where  $|\lambda| = |\mu| = 1$ .

As pointed out in [1], the converse of Theorem 3.1 is false as the example  $p(z) = (z + \frac{1}{2})(z + 3)$  shows. However, the following result for complex polynomials in the converse direction is valid. In fact, we can prove it in the quaternionic setting via the Hopf lemma, instead of the Lucas theorem.

**Theorem 3.2.** *Let  $P(q)$  be a slice regular polynomial of degree  $n$  such that  $P(1) = \|P\| = 1$ , and*

$$\max_{|q|=R} |P(q)| \leq \frac{1 + R^n}{2},$$

for  $1 < R < \delta$ , where  $\delta$  is any positive number. Then  $P$  does not have all its zeros within the unit Ball  $\mathbb{B}$ .

Now we recall some necessary definitions and properties from [4, 7, 9] in order to prove Theorem 3.2.

**Definition 3.3.** Let  $f, g : \mathbb{B} \rightarrow \mathbb{H}$  be two slice regular functions of the form

$$f(q) = \sum_{n=0}^{\infty} q^n a_n, \quad g(q) = \sum_{n=0}^{\infty} q^n b_n.$$

The regular product (or  $*$ -product) of  $f$  and  $g$  is the slice regular function defined by

$$f * g(q) = \sum_{n=0}^{\infty} q^n \left( \sum_{k=0}^n a_k b_{n-k} \right).$$

Notice that the  $*$ -product is associative and is not, in general, commutative. Its connection with the usual pointwise product is clarified by the following result.

**Proposition 3.4.** *Let  $f$  and  $g$  be slice regular on  $\mathbb{B}$ . Then for all  $q \in \mathbb{B}$ ,*

$$f * g(q) = \begin{cases} f(q)g(f(q)^{-1}qf(q)) & \text{if } f(q) \neq 0; \\ 0 & \text{if } f(q) = 0. \end{cases}$$

We remark that if  $q = x + yI$  and  $f(q) \neq 0$ , then  $f(q)^{-1}qf(q)$  has the same modulus and same real part as  $q$ . Therefore  $f(q)^{-1}qf(q)$  lies in the same 2-sphere  $x + y\mathbb{S}$  as  $q$ . Notice that a zero  $x_0 + y_0I$  of the function  $g$  is not necessarily a zero of  $f * g$ , but some element on the same sphere  $x_0 + y_0\mathbb{S}$  does. In particular, a real zero of  $g$  is still a zero of  $f * g$ . To present a characterization of the structure of the zero set of a regular function  $f$ , we need to introduce the *regular conjugate* of  $f$

$$f^c(q) = \sum_{n=0}^{\infty} q^n \bar{a}_n,$$

and the *symmetrization* of  $f$

$$f^s(q) = f * f^c(q) = f^c * f(q) = \sum_{n=0}^{\infty} q^n \left( \sum_{k=0}^n a_k \bar{a}_{n-k} \right).$$

**Proposition 3.5.** *Let  $f$  be a slice regular function on a symmetric slice domain  $\Omega$  and choose  $S = x + y\mathbb{S} \subset \Omega$ . The zeros of  $f$  in  $S$  are in one-to-one correspondence with those of  $f^c$ . Furthermore,  $f^s$  vanishes identically on  $S$  if and only if  $f^s$  has a zero in  $S$ , if and only if  $f$  has a zero in  $S$  (if and only if  $f^c$  has a zero in  $S$ ).*

We also need the following result.

**Proposition 3.6.** [17, Proposition 3.2] *Let  $f$  be a slice regular function on  $B = B(0, R)$ . For any sphere of the form  $x + y\mathbb{S}$  contained in  $B$  the following equality holds:*

$$\sup_{I \in \mathbb{S}} |f(x + yI)| = \sup_{I \in \mathbb{S}} |f^c(x + yI)|.$$

Now we can give the proof Theorem 3.2.

*Proof of theorem 3.2.* Let  $P$  be a slice regular polynomial of degree  $n$ . Consider its symmetrization  $P^s$  of the quaternionic polynomial  $P$  and notice that  $P^s$  is quaternionic polynomial with real coefficients. From Proposition 3.5, we see that  $P^s$  has all its zeros in  $\mathbb{B}$ , and hence that, from Lemma 3.7 below,

$$(3.1) \quad (P^s)'(1) > n.$$

It is easy to see that, by Proposition 3.4,

$$(3.2) \quad P^s(1) = 1.$$

From Propositions 3.4 and 3.6, it holds that, for  $1 < R < \delta$ ,

$$\max_{|q|=R} |P^s(q)| \leq \max_{|q|=R} |P(q)| \max_{|q|=R} |P^c(q)| = \max_{|q|=R} |P(q)|^2,$$

which implies that

$$(3.3) \quad \max_{|q|=R} |P^s(q)| \leq \left( \frac{1+R^n}{2} \right)^2 \leq \frac{1+R^{2n}}{2}.$$

Combining (3.2) with (3.3), we see that  $\|P^s\| = 1$ .

If  $P^s$  is constant, then the claim is trivial. Otherwise, the Hopf lemma [16, Lemma 15.3.7] shows that  $(P^s)'(1) > 0$ . Given any  $\epsilon > 0$ , sufficiently small, we have

$$|P^s(1+\epsilon) - P^s(1)| = P^s(1+\epsilon) - P^s(1) \leq \frac{1+\epsilon^{2n}}{2} - 1.$$

Hence

$$(P^s)'(1) \leq n,$$

which contradicts inequality (3.1). Now we complete the proof of theorem.  $\square$

To establish Theorem 3.2, Ankeny need the following lemma in terms of the Laguerre Theorem. Here we offer an elementary proof.

**Lemma 3.7.** *If*

$$p(z) = (z - z_1) \cdots (z - z_n)$$

where  $z_m \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  for  $m = 1, 2, \dots, n$ , then,

$$\left| \frac{p'(a)}{p(a)} \right| > \frac{n}{2}, \quad \forall a \in \partial\mathbb{D}.$$

*Proof.* Let  $a = e^{i\theta}$  for some  $\theta \in \mathbb{R}$ . Then,

$$\operatorname{Re} \frac{e^{i\theta}}{e^{i\theta} - z_m} > \frac{1}{2}$$

for  $z_m \in \mathbb{D}$ ,  $m = 1, 2, \dots, n$ . Hence,

$$\left| \frac{p'(a)}{p(a)} \right| \geq \operatorname{Re} \left( e^{i\theta} \frac{p'(e^{i\theta})}{p(e^{i\theta})} \right) = \sum_{m=1}^n \operatorname{Re} \frac{e^{i\theta}}{e^{i\theta} - z_m} > \frac{n}{2}.$$

$\square$

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